

# SOME PROPERTIES OF A CLASS OF ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

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ABSTRACT. We prove Schwarz-Pick type estimates and coefficient estimates for a class of functions induced by the elliptic partial differential operators. Then we apply these results to obtain a Landau type theorem.

## 1. INTRODUCTION AND MAIN RESULTS

Let  $\mathbb{C}$  be the complex plane. For  $a \in \mathbb{C}$ , let  $r > 0$  and  $\mathbb{D}(a, r) = \{z : |z - a| < r\}$ . In particular, we use  $\mathbb{D}_r$  to denote the disk  $\mathbb{D}(0, r)$  and  $\mathbb{D}$ , the open unit disk  $\mathbb{D}_1$ .

For a real  $2 \times 2$  matrix, we will consider the matrix norm  $\|A\| = \sup\{|Az| : |z| = 1\}$  and the matrix function  $l(A) = \inf\{|Az| : |z| = 1\}$ . For  $z = x + iy \in \mathbb{C}$  with  $x$  and  $y$  real, we denote the *complex differential operators*

$$\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) \text{ and } \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

If we denote the formal derivative of  $f = u + iv$  by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

then  $\|D_f\| = |f_z| + |f_{\bar{z}}|$  and  $l(D_f) = ||f_z| - |f_{\bar{z}}||$ , where  $u, v$  are real functions,  $f_z = \partial f / \partial z$  and  $f_{\bar{z}} = \partial f / \partial \bar{z}$ . Throughout this paper, we denote by  $C^n(\mathbb{D})$  the set of all  $n$ -times continuously differentiable complex-valued functions in  $\mathbb{D}$ , where  $n \in \{1, 2, \dots\}$ .

For  $\alpha \in \mathbb{R}$  and  $z \in \mathbb{D}$ , let

$$T_\alpha = -\frac{\alpha^2}{4}(1 - |z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1 - |z|^2)^{-\alpha-1} \left( z \frac{\partial}{\partial z} + \bar{z} \frac{\partial}{\partial \bar{z}} \right) + \frac{1}{4}(1 - |z|^2)^{-\alpha} \Delta$$

be the *second order elliptic partial differential operator*, where  $\Delta$  is the usual complex Laplacian operator

$$\Delta := 4 \frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

We consider the *Dirichlet boundary value problem* of distributional sense as follows

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$$(1.1) \quad \begin{cases} T_\alpha(f) = 0 & \text{in } \mathbb{D}, \\ f = f^* & \text{on } \partial\mathbb{D}. \end{cases}$$

Here, the boundary data  $f^* \in \mathfrak{D}'(\partial\mathbb{D})$  is a *distribution* on the boundary  $\partial\mathbb{D}$  of  $\mathbb{D}$ , and the boundary condition in (1.1) is interpreted in the distributional sense that  $f_r \rightarrow f^*$  in  $\mathfrak{D}'(\partial\mathbb{D})$  as  $r \rightarrow 1-$ , where

$$(1.2) \quad f_r(e^{i\theta}) = f(re^{i\theta}), \quad e^{i\theta} \in \partial\mathbb{D},$$

for  $r \in [0, 1)$  (see [24]).

In [24], Olofsson proved that, for parameter values  $\alpha > -1$ , a function  $f \in \mathcal{C}^2(\mathbb{D})$  satisfies (1.1) if and only if it has the form of a *Poisson type integral*

$$(1.3) \quad f(z) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-i\tau}) f^*(e^{i\tau}) d\tau, \quad \text{for } z \in \mathbb{D},$$

where

$$K_\alpha(z) = c_\alpha \frac{(1 - |z|^2)^{\alpha+1}}{|1 - z|^{\alpha+2}},$$

$c_\alpha = (\Gamma(\alpha/2 + 1))^2 / \Gamma(1 + \alpha)$  and  $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$  for  $s > 0$  is the standard Gamma function. If we take  $\alpha = 2(n-1)$ , then  $f$  is *polyharmonic* (or *n-harmonic*), where  $n \in \{1, 2, \dots\}$  (cf. [1, 2, 5, 16, 22, 23, 25, 26]). Furthermore, Borichev and Hedenmalm [5] proved that

$$(1 - |z|^2)^n \Delta^n = 4(1 - |z|^2) T_0 \circ 4(1 - |z|^2)^2 T_2 \circ \dots \circ 4(1 - |z|^2)^n T_{2(n-1)}.$$

In particular, if  $\alpha = 0$ , then  $f$  is harmonic.

For  $a, b, c \in \mathbb{R}$  with  $c \neq 0, -1, -2, \dots$ , the *hypergeometric* function is defined by the power series

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1,$$

where  $(a)_0 = 1$  and  $(a)_n = a(a+1) \cdots (a+n-1)$  for  $n = 1, 2, \dots$  are the *Pochhammer* symbols. Obviously, for  $n = 0, 1, 2, \dots$ ,  $(a)_n = \Gamma(a+n)/\Gamma(a)$ . In particular, for  $a, b, c > 0$  and  $a+b < c$ , we have (cf. [3, 4])

$$(1.4) \quad F(a, b; c; 1) = \lim_{x \rightarrow 1} F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} < \infty.$$

For  $\alpha = 0$ , Heinz [17] and Colonna [14] proved the following Schwarz-Pick type estimates on planar harmonic functions, which are the following. For the related discussions on this topic, see [6, 9, 13, 19, 27].

**Theorem A.** ([17, Lemma]) *Let  $f$  be a harmonic function of  $\mathbb{D}$  into  $\mathbb{D}$  with  $f(0) = 0$ . Then, for  $z \in \mathbb{D}$ ,*

$$|f(z)| \leq \frac{4}{\pi} \arctan |z|.$$

*This estimate is sharp.*

**Theorem B.** ([14, Theorems 3 and 4]) *Let  $f$  be a harmonic function of  $\mathbb{D}$  into  $\mathbb{D}$ . Then, for  $z \in \mathbb{D}$ ,*

$$\|D_f(z)\| \leq \frac{4}{\pi} \frac{1}{1 - |z|^2}.$$

*This estimate is sharp, and all the extremal functions are*

$$f(z) = \frac{2\gamma}{\pi} \arg \left( \frac{1 + \psi(z)}{1 - \psi(z)} \right),$$

*where  $|\gamma| = 1$  and  $\psi$  is a conformal automorphism of  $\mathbb{D}$ .*

For  $\alpha > -1$ , we establish the following Schwarz-Pick type estimate on the solutions to (1.1).

**Theorem 1.** *For  $\alpha > -1$ , let  $f \in \mathcal{C}^2(\mathbb{D})$  satisfy (1.1) and  $\sup_{z \in \mathbb{D}} |f(z)| \leq M$ , where  $M$  is a positive constant. Then, for  $z \in \mathbb{D}$ ,*

$$(1.5) \quad \left| f(z) - \frac{(1 - |z|)^{\alpha+1}}{1 + |z|} f(0) \right| \leq M \left[ \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-it}) dt - \frac{(1 - |z|)^{\alpha+1}}{1 + |z|} K_\alpha(0) \right]$$

*and*

$$(1.6) \quad \|D_f(z)\| \leq \frac{M \mathcal{M}_\alpha(|z|) [2 + \alpha + (4 + 3\alpha)|z|]}{1 - |z|^2} \leq \frac{M [2 + \alpha + (4 + 3\alpha)|z|]}{1 - |z|^2},$$

*where*

$$(1.7) \quad \mathcal{M}_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\epsilon}) d\epsilon = \frac{[\Gamma(1 + \frac{\alpha}{2})]^2}{\Gamma(1 + \alpha)} F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; r^2\right), \quad r \in [0, 1).$$

Let  $f$  be a harmonic mapping of  $\overline{\mathbb{D}}$  onto  $\overline{\mathbb{D}}$  with  $f(0) = 0$ . In [17], Heinz showed that, for  $\theta \in [0, 2\pi]$ ,

$$\|D_f(e^{i\theta})\| \geq \frac{2}{\pi}.$$

For the extensive discussion on the Heinz's inequality for real harmonic functions in high dimension, see [18].

By using Theorem 1, we get a Heinz type inequality on  $\partial\mathbb{D}$  as follows.

**Theorem 2.** *For  $\alpha \geq 0$ , let  $f \in \mathcal{C}^2(\overline{\mathbb{D}})$  satisfying (1.1). Suppose that  $f(0) = 0$ ,  $f(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$  and  $f(\partial\mathbb{D}) = \partial\mathbb{D}$ .*

(a) *If  $\alpha = 0$ , then, for  $\theta \in [0, 2\pi]$ ,*

$$\|D_f(e^{i\theta})\| \geq \frac{2}{\pi};$$

(b) *If  $\alpha > 0$ , then, for  $\theta \in [0, 2\pi]$ ,*

$$\|D_f(e^{i\theta})\| \geq \lim_{r \rightarrow 1^-} \frac{d}{dr} \mathcal{M}_\alpha(r) = \frac{\alpha}{2},$$

*where  $\mathcal{M}_\alpha(r)$  is given by (1.7).*

The following result is the homogeneous expansion of solutions to (1.1).

**Theorem C.** ([24, Theorem 2.2]) *Let  $\alpha \in \mathbb{R}$  and  $f \in \mathcal{C}^2(\mathbb{D})$ . Then  $f$  satisfies (1.1) if and only if it has a series expansion of the form*

$$(1.8) \quad f(z) = \sum_{k=0}^{\infty} c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2\right) z^k \\ + \sum_{k=1}^{\infty} c_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2\right) \bar{z}^k, \quad z \in \mathbb{D},$$

for some sequence  $\{c_k\}_{k=-\infty}^{\infty}$  of complex numbers satisfying

$$(1.9) \quad \lim_{|k| \rightarrow \infty} \sup |c_k|^{\frac{1}{|k|}} \leq 1.$$

In particular, the expansion (1.8), subject to (1.9), converges in  $\mathcal{C}^\infty(\mathbb{D})$ , and every solution  $f$  of (1.1) is  $\mathcal{C}^\infty$ -smooth in  $\mathbb{D}$ .

For  $\alpha = 0$ , there are numerous discussions on coefficient estimates of harmonic mappings in the literature, see for example [8, 9, 10, 12, 15, 21, 29]. We investigate the problem of coefficient estimates on the solutions to (1.1) as follows.

**Theorem 3.** *For  $\alpha > -1$ , let  $f \in \mathcal{C}^2(\mathbb{D})$  be a solution to (1.1) with the series expansion of the form (1.8) and  $\sup_{z \in \mathbb{D}} |f(z)| \leq M$ , where  $M$  is a positive constant. Then, for  $k \in \{1, 2, \dots\}$ ,*

$$(1.10) \quad \left| c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; 1\right) \right| + \left| c_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; 1\right) \right| \leq \frac{4M}{\pi}$$

and

$$\left| c_0 F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1\right) \right| \leq M.$$

In particular, if  $\alpha = 0$ , then the estimate of (1.10) is sharp and all the extreme functions are

$$f_k(z) = \frac{2\varepsilon M}{\pi} \operatorname{Im} \left( \log \frac{1 + \vartheta z^k}{1 - \vartheta z^k} \right),$$

where  $|\varepsilon| = |\vartheta| = 1$ .

The following result easily follows from Theorem 3 and [24, Proposition 1.4].

**Corollary 1.1.** *For  $\alpha > -1$ , let  $f \in \mathcal{C}^2(\mathbb{D})$  be a solution to (1.1) with the series expansion of the form (1.8) and  $\sup_{z \in \mathbb{D}} |f(z)| \leq M$ , where  $M$  is a positive constant. Then, for  $k \in \{1, 2, \dots\}$ ,*

$$|c_k| + |c_{-k}| \leq \frac{4M\Gamma\left(1 + \frac{\alpha}{2}\right)\Gamma\left(k + 1 + \frac{\alpha}{2}\right)}{k!\Gamma(\alpha + 1)\pi}.$$

For  $p \in (0, \infty]$ , the *Hardy space*  $\mathcal{H}^p$  consists of those functions  $f : \mathbb{D} \rightarrow \mathbb{C}$  such that  $f$  is measurable,  $M_p(r, f)$  exists for all  $r \in (0, 1)$  and  $\|f\|_p < \infty$ , where

$$\|f\|_p = \begin{cases} \sup_{0 < r < 1} M_p(r, f), & \text{if } p \in (0, \infty), \\ \sup_{z \in \mathbb{D}} |f(z)|, & \text{if } p = \infty, \end{cases}$$

and

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

It is not difficult to know that all bounded measurable functions belong to  $\mathcal{H}^p$ .

The classical theorem of Landau shows that there is a  $\rho = \frac{1}{M + \sqrt{M^2 - 1}}$  such that every function  $f$ , analytic in  $\mathbb{D}$  with  $f(0) = f'(0) - 1 = 0$  and  $|f(z)| < M$  in  $\mathbb{D}$ , is univalent in the disk  $\mathbb{D}_\rho$  and in addition, the range  $f(\mathbb{D}_\rho)$  contains a disk of radius  $M\rho^2$  (see [20]), where  $M \geq 1$  is a constant. Recently, many authors considered Landau type theorem for planar harmonic mappings (see [7, 8, 9, 10]), biharmonic mappings (see [1]) and polyharmonic mappings (see [12]).

The last main aim of this paper is to establish a Landau type theorem for a more general class of functions. Moreover, we do not need the condition of bounded functions as in the classical Landau Theorem. Applying Theorems 1 and 3, we get the following Landau type theorem for a class of functions  $f \in \mathcal{H}^p$  satisfying (1.1).

**Theorem 4.** *For  $\alpha \in (-1, 0]$ , let  $f \in \mathcal{C}^2(\mathbb{D})$  be a solution to (1.1) satisfying  $f(0) = |J_f(0)| - \lambda = 0$  and  $f \in \mathcal{H}^p$ , where  $\lambda$  is a positive constant and  $J_f$  is the Jacobian of  $f$ . Then  $f$  is univalent in  $\mathbb{D}_{\gamma_0\rho_0}$ , where  $\rho_0$  satisfies the following equation*

$$\frac{\lambda}{M^*(2 + \alpha)} - \frac{4M^*\rho_0}{\pi} \left[ \frac{2 - \rho_0}{(1 - \rho_0)^2} + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right] = 0,$$

where  $\mu(\gamma) = (1 + \gamma)^{\frac{\alpha+1}{p}} / [\gamma(1 - \gamma)^{\frac{1}{p}}]$ ,  $\mu(\gamma_0) = \min_{0 < \gamma < 1} \mu(\gamma)$  and  $M^* = c_\alpha^{\frac{1}{p}} \|f\|_p \mu(\gamma_0)$ .

Moreover,  $f(\mathbb{D}_{\gamma_0\rho_0})$  contains a univalent disk  $\mathbb{D}_{\gamma_0 R_0}$  with

$$R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\lambda}{M^*(2 + \alpha)} - \frac{M^*\rho_0(2 - \rho_0)}{\pi(1 - \rho_0)^2} \right].$$

We remark that Theorem 4 is a generalization of [7, Theorem 2] and [11, Theorem 5].

The proofs of Theorems 1, 2 and 3 will be presented in Section 2, and the proof of Theorem 4 will be given in Section 3.

## 2. SCHWARZ-PICK TYPE ESTIMATES AND COEFFICIENT ESTIMATES

**Proof of Theorem 1.** We first prove (1.5). By the assumption, we see that  $f_r \rightarrow f^*$  in  $\mathfrak{D}'(\partial\mathbb{D})$  as  $r \rightarrow 1-$ , where  $f_r$  is given by (1.2) for  $r \in [0, 1)$ . By (1.3), for  $z = re^{i\theta} \in \mathbb{D}$ , we have

$$\begin{aligned}
& \left| f(z) - \frac{(1-|z|)^{\alpha+1}}{1+|z|} f(0) \right| \\
&= \frac{1}{2\pi} \left| \int_0^{2\pi} K_\alpha(ze^{-it}) f^*(e^{it}) dt - \frac{(1-|z|)^{\alpha+1}}{1+|z|} \int_0^{2\pi} K_\alpha(0) f^*(e^{it}) dt \right| \\
&= \frac{1}{2\pi} \left| \int_0^{2\pi} \left( K_\alpha(ze^{-it}) - \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0) \right) f^*(e^{it}) dt \right| \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \left( K_\alpha(ze^{-it}) - \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0) \right) |f^*(e^{it})| dt \\
&\leq M \left[ \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-it}) dt - \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0) \right].
\end{aligned}$$

Next we prove (1.6). By the proof of [24, Theorem 3.1], we observe that

$$(2.1) \quad \mathcal{M}_\alpha(r) = \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\epsilon}) d\epsilon = \frac{[\Gamma(1+\frac{\alpha}{2})]^2}{\Gamma(1+\alpha)} F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; r^2\right)$$

and  $\mathcal{M}_\alpha(r)$  is increasing on  $r \in [0, 1)$  with

$$\lim_{r \rightarrow 1^-} \mathcal{M}_\alpha(r) = 1.$$

By elementary calculations, for  $z \in \mathbb{D}$ , we have

$$\frac{\partial}{\partial z} K_\alpha(ze^{-it}) = c_\alpha \frac{(1-|z|^2)^\alpha \left[ \left(1+\frac{\alpha}{2}\right) e^{-it} (1-\bar{z}e^{it})(1-|z|^2) - (\alpha+1)\bar{z}|1-ze^{-it}|^2 \right]}{|1-ze^{-it}|^{4+\alpha}}$$

and

$$\frac{\partial}{\partial \bar{z}} K_\alpha(ze^{-it}) = c_\alpha \frac{(1-|z|^2)^\alpha \left[ \left(1+\frac{\alpha}{2}\right) e^{it} (1-ze^{-it})(1-|z|^2) - (\alpha+1)z|1-ze^{-it}|^2 \right]}{|1-ze^{-it}|^{4+\alpha}},$$

which, together with (1.3) and (2.1), imply that

$$\begin{aligned}
\|D_f(z)\| &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial z} K_\alpha(ze^{-it}) f^*(e^{it}) dt \right| + \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial}{\partial \bar{z}} K_\alpha(ze^{-it}) f^*(e^{it}) dt \right| \\
&\leq \frac{Mc_\alpha}{\pi} \int_0^{2\pi} \frac{(1-|z|^2)^\alpha \left[ (1+\alpha)|z||1-ze^{-it}|^2 + \left(1+\frac{\alpha}{2}\right)|1-ze^{-it}||1-|z|^2| \right]}{|1-ze^{-it}|^{4+\alpha}} dt
\end{aligned}$$

$$\begin{aligned}
&= \frac{Mc_\alpha}{\pi} \left[ \int_0^{2\pi} \frac{(1+\alpha)|z|(1-|z|^2)^\alpha}{|1-ze^{-it}|^{2+\alpha}} dt + \int_0^{2\pi} \frac{(1+\frac{\alpha}{2})(1-|z|^2)^{\alpha+1}}{|1-ze^{-it}|^{3+\alpha}} dt \right] \\
&\leq \frac{Mc_\alpha}{\pi} \left[ \frac{(1+\alpha)|z|}{(1-|z|^2)} \int_0^{2\pi} \frac{(1-|z|^2)^{\alpha+1}}{|1-ze^{-it}|^{2+\alpha}} dt + \frac{(1+\frac{\alpha}{2})}{(1-|z|)} \int_0^{2\pi} \frac{(1-|z|^2)^{\alpha+1}}{|1-ze^{-it}|^{2+\alpha}} dt \right] \\
&= \left[ \frac{2M(1+\alpha)|z|}{1-|z|^2} + \frac{M(2+\alpha)}{1-|z|} \right] \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-it}) dt \\
&= \left[ \frac{2M(1+\alpha)|z|}{1-|z|^2} + \frac{M(2+\alpha)}{1-|z|} \right] \mathcal{M}_\alpha(|z|) \\
&= \frac{M\mathcal{M}_\alpha(|z|)[2+\alpha+(4+3\alpha)|z|]}{1-|z|^2} \\
&\leq \frac{M[2+\alpha+(4+3\alpha)|z|]}{1-|z|^2}.
\end{aligned}$$

The proof of this theorem is complete.  $\square$

**Proof of Theorem 2.** Since (a) easily follows from the inequality (15) in [17], we only need to prove (b). Let  $\alpha > 0$ . By Theorem 1 (1.5), we have

$$\begin{aligned}
(2.2) \quad \frac{|f(e^{i\theta}) - f(re^{i\theta})|}{1-r} &\geq \frac{1 - |f(re^{i\theta})|}{1-r} \\
&\geq \frac{1 - \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(ze^{-it}) dt + \frac{(1-|z|)^{\alpha+1}}{1+|z|} K_\alpha(0)}{1-r}.
\end{aligned}$$

where  $z = re^{i\theta} \in \mathbb{D}$  and  $\theta \in [0, 2\pi)$ .

Applying [24, Theorem 3.1], we get

$$\frac{1}{2\pi} \lim_{|z| \rightarrow 1^-} \int_0^{2\pi} K_\alpha(ze^{-it}) dt = \lim_{r \rightarrow 1^-} \mathcal{M}_\alpha(r) = 1,$$

which, together with L'Hopital's rule and (2.2), yield that

$$\begin{aligned}
\|D_f(e^{i\theta})\| &\geq \left( \left| \frac{\partial f(re^{i\theta})}{\partial r} \right| \right)_{r=1} \\
&= \lim_{r \rightarrow 1^-} \frac{|f(e^{i\theta}) - f(re^{i\theta})|}{1-r} \\
&\geq \lim_{r \rightarrow 1^-} \frac{1 - \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i(\theta-t)}) dt + \frac{(1-r)^{\alpha+1}}{1+r} K_\alpha(0)}{1-r} \\
&= \lim_{r \rightarrow 1^-} \frac{1 - \frac{1}{2\pi} \int_0^{2\pi} K_\alpha(re^{i\eta}) d\eta + \frac{(1-r)^{\alpha+1}}{1+r} K_\alpha(0)}{1-r}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2\pi} \lim_{r \rightarrow 1^-} \frac{d}{dr} \int_0^{2\pi} K_\alpha(re^{i\eta}) d\eta \\
&= \lim_{r \rightarrow 1^-} \frac{d}{dr} \mathcal{M}_\alpha(r).
\end{aligned}$$

where  $\mathcal{M}_\alpha(r)$  is given by (1.7). It follows from the proof of [24, Theorem 3.1] that

$$\mathcal{M}_\alpha(r) = \frac{[\Gamma(1 + \frac{\alpha}{2})]^2}{\Gamma(1 + \alpha)} F\left(-\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; r^2\right) = \frac{[\Gamma(1 + \frac{\alpha}{2})]^2}{\Gamma(1 + \alpha)} \sum_{n=0}^{\infty} \frac{[(\frac{\alpha}{2})_n]^2}{(n!)^2} r^{2n},$$

which yields that

$$\frac{d}{dr} \mathcal{M}_\alpha(r) = \frac{\alpha^2}{2} \frac{[\Gamma(1 + \frac{\alpha}{2})]^2}{\Gamma(1 + \alpha)} r F\left(1 - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; 2; r^2\right),$$

where  $r \in (0, 1)$ . By (1.4), for  $\alpha > 0$ , we see that

$$\begin{aligned}
\lim_{r \rightarrow 1^-} \frac{d}{dr} \mathcal{M}_\alpha(r) &= \frac{\alpha^2}{2} \frac{[\Gamma(1 + \frac{\alpha}{2})]^2}{\Gamma(1 + \alpha)} F\left(1 - \frac{\alpha}{2}, 1 - \frac{\alpha}{2}; 2; 1\right) \\
&= \frac{\alpha^2}{2} \frac{[\Gamma(1 + \frac{\alpha}{2})]^2}{\Gamma(1 + \alpha)} \frac{\Gamma(2)\Gamma(\alpha)}{[\Gamma(1 + \frac{\alpha}{2})]^2} \\
&= \frac{\alpha^2}{2} \frac{\Gamma(\alpha)}{\Gamma(1 + \alpha)} = \frac{\alpha}{2}.
\end{aligned}$$

Therefore, for  $\theta \in [0, 2\pi]$ ,

$$\|D_f(e^{i\theta})\| \geq \lim_{r \rightarrow 1^-} \frac{d}{dr} \mathcal{M}_\alpha(r) = \frac{\alpha}{2},$$

where  $\alpha > 0$ . The proof of this theorem is complete.  $\square$

**Proof of Theorem 3.** For  $r \in [0, 1)$ , let

$$A_k(r, \alpha) = c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; r^2\right)$$

and

$$B_k(r, \alpha) = c_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k + 1; r^2\right),$$

where  $r = |z|$ . Then

$$A_k(r, \alpha) r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z) e^{-ik\theta} d\theta$$

and

$$B_k(r, \alpha) r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z) e^{ik\theta} d\theta,$$

which imply that

$$(2.3) \quad |A_k(r, \alpha)| r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z) e^{-ik\theta} e^{-i \arg A_k(r, \alpha)} d\theta$$



and

$$(2.4) \quad |B_k(r, \alpha)|r^k = \frac{1}{2\pi} \int_0^{2\pi} f(z) e^{ik\theta} e^{-i \arg B_k(r, \alpha)} d\theta,$$

where  $A_k(r, \alpha) = |A_k(r, \alpha)|e^{i \arg A_k(r, \alpha)}$ ,  $B_k(r, \alpha) = |B_k(r, \alpha)|e^{i \arg B_k(r, \alpha)}$  and  $z = re^{i\theta}$ . By (2.3), (2.4) and [13, Lemma 1], we have

$$(2.5) \quad \begin{aligned} & |(|A_k(r, \alpha)| + |B_k(r, \alpha)|)r^k| \\ &= \left| \frac{1}{2\pi} \int_0^{2\pi} f(z) \left[ e^{-i(k\theta + \arg A_k(r, \alpha))} + e^{i(k\theta - \arg B_k(r, \alpha))} \right] d\theta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z)| \left| e^{-i(k\theta + \arg A_k(r, \alpha))} + e^{i(k\theta - \arg B_k(r, \alpha))} \right| d\theta \\ &\leq \frac{M}{2\pi} \int_0^{2\pi} \left| e^{-i(k\theta + \arg A_k(r, \alpha))} + e^{i(k\theta - \arg B_k(r, \alpha))} \right| d\theta \\ &= \frac{M}{2\pi} \int_0^{2\pi} \left| 1 + e^{i(2k\theta + \arg A_k(r, \alpha) - \arg B_k(r, \alpha))} \right| d\theta \\ &= \frac{M}{\pi} \int_0^{2\pi} \left| \cos \left( k\theta + \frac{\arg A_k(r, \alpha) - \arg B_k(r, \alpha)}{2} \right) \right| d\theta \\ &= \frac{4M}{\pi}. \end{aligned}$$

By letting  $r \rightarrow 1-$  on (2.5), we obtain

$$|A_k(1, \alpha)| + |B_k(1, \alpha)| \leq \frac{4M}{\pi}.$$

On the other hand, for  $k = 0$ , we have

$$(2.6) \quad \begin{aligned} \frac{1}{2\pi} \lim_{r \rightarrow 1-} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta &= \left| c_0 F \left( -\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1 \right) \right|^2 \\ &\quad + \sum_{k=1}^{\infty} \left( \left| c_k F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; 1 \right) \right|^2 \right. \\ &\quad \left. + \left| c_{-k} F \left( -\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; 1 \right) \right|^2 \right) \\ &\leq M^2, \end{aligned}$$

where  $r \in [0, 1)$ . It follows from (2.6) that

$$\left| c_0 F \left( -\frac{\alpha}{2}, -\frac{\alpha}{2}; 1; 1 \right) \right| \leq M.$$

If  $\alpha = 0$ , then the sharpness part follows from [12, Lemma 1]. The proof of this theorem is complete.  $\square$

## 3. THE LANDAU TYPE THEOREM

**Lemma 1.** *For  $x \in [0, 1)$ , let*

$$\varphi(x) = \frac{\delta}{M(2+\alpha)} - \frac{4Mx}{\pi} \left[ \frac{(2-x)}{(1-x)^2} + \frac{2x}{(1-x)(1-x^2)^2} \right],$$

where  $\alpha > -2$ ,  $\delta > 0$  and  $M > 0$  are constant. Then  $\varphi$  is strictly decreasing and there is an unique  $x_0 \in (0, 1)$  such that  $\varphi(x_0) = 0$ .

*Proof.* For  $x \in [0, 1)$ , let

$$f_1(x) = \frac{4M}{\pi} \frac{x(2-x)}{(1-x)^2} \text{ and } f_2(x) = \frac{4M}{\pi} \frac{2x^2}{(1-x)(1-x^2)^2}.$$

Since, for  $x \in [0, 1)$ ,

$$f_1'(x) = \frac{8M}{\pi} \frac{1}{(1-x)^3} > 0,$$

we see that  $f_1$  is continuous and strictly increasing in  $[0, 1)$ . We observe that  $f_2$  is also continuous and strictly increasing in  $[0, 1)$ . Then

$$\varphi(x) = \frac{\delta}{M(2+\alpha)} - f_1(x) - f_2(x)$$

is continuous and strictly decreasing in  $[0, 1)$ , which, together with

$$\lim_{x \rightarrow 0} \varphi(x) = \frac{\delta}{M(2+\alpha)} > 0 \text{ and } \lim_{x \rightarrow 1^-} \varphi(x) = -\infty,$$

imply that there is an unique  $x_0 \in (0, 1)$  such that  $\varphi(x_0) = 0$ .  $\square$

**Lemma 2.** *For  $\alpha \in (-1, 0]$ , let  $f \in \mathcal{C}^2(\mathbb{D})$  be a solution to (1.1) satisfying  $f(0) = |J_f(0)| - \beta = 0$  and  $\sup_{z \in \mathbb{D}} |f(z)| \leq M$ , where  $M, \beta$  are positive constants and  $J_f$  is the Jacobian of  $f$ . Then  $f$  is univalent in  $\mathbb{D}_{\rho_0}$ , where  $\rho_0$  satisfies the following equation*

$$\frac{\beta}{M(2+\alpha)} - \frac{4M\rho_0}{\pi} \left[ \frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2} \right] = 0.$$

Moreover,  $f(\mathbb{D}_{\rho_0})$  contains a univalent disk  $\mathbb{D}_{R_0}$  with

$$R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\beta}{M(2+\alpha)} - \frac{M\rho_0(2-\rho_0)}{\pi(1-\rho_0)^2} \right].$$

*Proof.* By Theorem C, we can assume that

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2\right) z^k \\ &\quad + \sum_{k=1}^{\infty} c_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2\right) \bar{z}^k, \quad z \in \mathbb{D}, \end{aligned}$$

for some sequence  $\{c_k\}_{k=-\infty}^{\infty}$  of complex numbers satisfying

$$\lim_{|k| \rightarrow \infty} \sup |c_k|^{\frac{1}{|k|}} \leq 1.$$

For  $\alpha \in (-1, 0]$ , by [24, Proposition 1.4], we observe that

$$F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; r^2\right) = \sum_{n=0}^{\infty} \frac{\left(-\frac{\alpha}{2}\right)_n \left(k - \frac{\alpha}{2}\right)_n r^{2n}}{(k+1)_n n!} \geq 0$$

is bounded and increasing on  $r \in [0, 1)$ , which imply that

$$\begin{aligned} (3.1) \quad & (|c_k| + |c_{-k}|) F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; r^2\right) \\ & \leq (|c_k| + |c_{-k}|) F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; 1\right) \\ & \leq \frac{4M}{\pi}, \end{aligned}$$

where  $r = |z|$  and  $k \in \{1, 2, \dots\}$ .

By (3.1) and Theorem 3, we see that, for each  $k \in \{1, 2, \dots\}$ ,

$$(3.2) \quad (|c_k| + |c_{-k}|) \frac{\left(-\frac{\alpha}{2}\right)_n \left(k - \frac{\alpha}{2}\right)_n}{(k+1)_n} \frac{1}{n!} \leq \frac{4M}{\pi},$$

where  $n \in \{1, 2, \dots\}$ .

Since  $c_0 = f(0) = 0$ , we see that

$$\begin{aligned} (3.3) \quad f_z(z) - f_z(0) &= \sum_{k=2}^{\infty} k c_k F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) z^{k-1} \\ &+ \sum_{k=1}^{\infty} c_k \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) z^k \bar{z} \\ &+ \sum_{k=1}^{\infty} c_{-k} \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) \bar{z}^{k+1} \end{aligned}$$

and

$$\begin{aligned} (3.4) \quad f_{\bar{z}}(z) - f_{\bar{z}}(0) &= \sum_{k=2}^{\infty} k c_{-k} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) \bar{z}^{k-1} \\ &+ \sum_{k=1}^{\infty} c_k \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) z^{k+1} \\ &+ \sum_{k=1}^{\infty} c_{-k} \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) \bar{z}^k z, \end{aligned}$$

where  $w = |z|^2$ .

Applying (3.2), (3.3) and (3.4), we obtain

$$\begin{aligned}
(3.5) \quad & |f_z(z) - f_z(0)| + |f_{\bar{z}}(z) - f_{\bar{z}}(0)| \\
& \leq \sum_{k=2}^{\infty} k(|c_k| + |c_{-k}|) F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) |z|^{k-1} \\
& \quad + 2 \sum_{k=1}^{\infty} (|c_k| + |c_{-k}|) \frac{d}{dw} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; w\right) |z|^{k+1} \\
& \leq \frac{4M}{\pi} \sum_{k=2}^{\infty} k|z|^{k-1} + 2 \sum_{k=1}^{\infty} \left[ \frac{4M}{\pi} \sum_{n=1}^{\infty} n|z|^{2(n-1)} \right] |z|^{k+1} \\
& = \frac{4M}{\pi} \frac{|z|(2-|z|)}{(1-|z|)^2} + \frac{8M}{\pi} \sum_{k=1}^{\infty} \frac{|z|^{k+1}}{(1-|z|^2)^2} \\
& = \frac{4M}{\pi} \frac{|z|(2-|z|)}{(1-|z|)^2} + \frac{8M}{\pi} \frac{|z|^2}{(1-|z|)(1-|z|^2)^2}.
\end{aligned}$$

Applying Theorem 1 (1.6), we get

$$\beta = |J_f(0)| = |\det D_f(0)| = \|D_f(0)\| l(D_f(0)) \leq M(2+\alpha) l(D_f(0)),$$

which gives that

$$(3.6) \quad l(D_f(0)) \geq \frac{\beta}{M(2+\alpha)}.$$

In order to prove the univalence of  $f$  in  $\mathbb{D}_{\rho_0}$ , we choose two distinct points  $z_1, z_2 \in \mathbb{D}_{\rho_0}$  and let  $[z_1, z_2]$  denote the segment from  $z_1$  to  $z_2$  with the endpoints  $z_1$  and  $z_2$ , where  $\rho_0$  satisfies the following equation

$$\frac{\beta}{M(2+\alpha)} - \frac{4M\rho_0}{\pi} \left[ \frac{2-\rho_0}{(1-\rho_0)^2} + \frac{2\rho_0}{(1-\rho_0)(1-\rho_0^2)^2} \right] = 0.$$

By (3.5), (3.6) and Lemma 1, we have

$$\begin{aligned}
|f(z_2) - f(z_1)| &= \left| \int_{[z_1, z_2]} f_z(z) dz + f_{\bar{z}}(z) d\bar{z} \right| \\
&= \left| \int_{[z_1, z_2]} f_z(0) dz + f_{\bar{z}}(0) d\bar{z} \right| \\
&\quad - \left| \int_{[z_1, z_2]} (f_z(z) - f_z(0)) dz + (f_{\bar{z}}(z) - f_{\bar{z}}(0)) d\bar{z} \right| \\
&\geq l(D_f)(0) |z_2 - z_1| \\
&\quad - \int_{[z_1, z_2]} (|f_z(z) - f_z(0)| + |f_{\bar{z}}(z) - f_{\bar{z}}(0)|) |dz|
\end{aligned}$$

$$\begin{aligned}
&> |z_2 - z_1| \left\{ \frac{\beta}{M(2 + \alpha)} \right. \\
&\quad \left. - \frac{4M\rho_0}{\pi} \left[ \frac{2 - \rho_0}{(1 - \rho_0)^2} + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right] \right\} \\
&= 0.
\end{aligned}$$

Thus,  $f(z_2) \neq f(z_1)$ . The univalence of  $f$  follows from the arbitrariness of  $z_1$  and  $z_2$ . This implies that  $f$  is univalent in  $\mathbb{D}_{\rho_0}$ .

Now, for any  $\zeta' = \rho_0 e^{i\theta} \in \partial\mathbb{D}_{\rho_0}$ , we obtain that

$$\begin{aligned}
|f(\zeta') - f(0)| &= \left| \int_{[0, \zeta']} f_z(z) dz + f_{\bar{z}}(z) d\bar{z} \right| \\
&= \left| \int_{[0, \zeta']} f_z(0) dz + f_{\bar{z}}(0) d\bar{z} \right| \\
&\quad - \left| \int_{[0, \zeta']} (f_z(z) - f_z(0)) dz + (f_{\bar{z}}(z) - f_{\bar{z}}(0)) d\bar{z} \right| \\
&\geq l(D_f)(0)\rho_0 - \int_{[0, \zeta']} (|f_z(z) - f_z(0)| + |f_{\bar{z}}(z) - f_{\bar{z}}(0)|) |dz| \\
&\geq l(D_f)(0)\rho_0 - \frac{4M\rho_0^2}{\pi} \int_0^1 \left[ \frac{t(2 - \rho_0 t)}{(1 - \rho_0 t)^2} + \frac{2\rho_0 t^2}{(1 - \rho_0 t)(1 - \rho_0^2 t^2)^2} \right] dt \\
&\geq \frac{\beta\rho_0}{M(2 + \alpha)} - \frac{4M\rho_0^2}{\pi} \left[ \frac{(2 - \rho_0)}{(1 - \rho_0)^2} \int_0^1 t dt \right. \\
&\quad \left. + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \int_0^1 t^2 dt \right] \\
&= \rho_0 \left\{ \frac{\beta}{M(2 + \alpha)} - \frac{4M\rho_0}{\pi} \left[ \frac{2 - \rho_0}{2(1 - \rho_0)^2} + \frac{2\rho_0}{3(1 - \rho_0)(1 - \rho_0^2)^2} \right] \right\} \\
&= \frac{2\rho_0}{3} \left[ \frac{\beta}{M(2 + \alpha)} - \frac{M\rho_0(2 - \rho_0)}{\pi(1 - \rho_0)^2} \right].
\end{aligned}$$

Hence  $f(\mathbb{D}_{\rho_0})$  contains a univalent disk  $\mathbb{D}_{R_0}$  with

$$R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\beta}{M(2 + \alpha)} - \frac{M\rho_0(2 - \rho_0)}{\pi(1 - \rho_0)^2} \right].$$

The proof of this lemma is complete.  $\square$

Let us recall the following result which is referred to as *Jensen's inequality* (cf. [28]).

**Lemma D.** *Let  $(\Omega, A, \mu)$  be a measure space such that  $\mu(\Omega) = 1$ . If  $g$  is a real-valued function that is  $\mu$ -integrable, and if  $\chi$  is a convex function on the real line, then*

$$\chi\left(\int_{\Omega} g d\mu\right) \leq \int_{\Omega} \chi \circ g d\mu.$$

**Proof of Theorem 4.** For  $z \in \mathbb{D}_r$ , we have

$$f(z) = \frac{c_{\alpha}}{2\pi r^{\alpha}} \int_0^{2\pi} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} f(re^{it}) dt,$$

where  $r \in (0, 1)$ . Let

$$\phi_z(r) = \frac{c_{\alpha}}{2\pi r^{\alpha}} \int_0^{2\pi} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} dt,$$

where  $z \in \mathbb{D}_r$ . Applying [24, Theorem 3.1], we see that, for  $z \in \mathbb{D}$ ,

$$(3.7) \quad \phi_z(1) \leq \lim_{|z| \rightarrow 1-} \phi_z(1) = 1.$$

By using Jensen's inequality (see Lemma D), for  $p \geq 1$ , we get

$$\begin{aligned} \left| \frac{f(z)}{\phi_z(r)} \right|^p &= \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{c_{\alpha}}{r^{\alpha} \phi_z(r)} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} f(re^{it}) dt \right|^p \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{c_{\alpha}}{r^{\alpha} \phi_z(r)} \frac{(r^2 - |z|^2)^{\alpha+1}}{|r - ze^{-it}|^{2+\alpha}} |f(re^{it})|^p dt \\ &\leq \frac{c_{\alpha}}{r^{\alpha} \phi_z(r)} \frac{(r^2 - |z|^2)^{\alpha+1}}{(r - |z|)^{2+\alpha}} \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right) \\ &\leq \frac{c_{\alpha} \|f\|_p^p (r + |z|)^{\alpha+1}}{r^{\alpha} \phi_z(r) (r - |z|)}, \end{aligned}$$

which implies that

$$|f(z)| \leq \left[ \frac{c_{\alpha} \|f\|_p^p (\phi_z(r))^{p-1}}{r^{\alpha}} \right]^{\frac{1}{p}} \frac{(r + |z|)^{\frac{\alpha+1}{p}}}{(r - |z|)^{\frac{1}{p}}},$$

where  $z \in \mathbb{D}_r$ . By letting  $r \rightarrow 1-$  and (3.7), for  $z \in \mathbb{D}$ , we have

$$(3.8) \quad |f(z)| \leq \left[ c_{\alpha} \|f\|_p^p (\phi_z(1))^{p-1} \right]^{\frac{1}{p}} \frac{(1 + |z|)^{\frac{\alpha+1}{p}}}{(1 - |z|)^{\frac{1}{p}}} \leq c_{\alpha}^{\frac{1}{p}} \|f\|_p \frac{(1 + |z|)^{\frac{\alpha+1}{p}}}{(1 - |z|)^{\frac{1}{p}}}.$$

For  $\zeta \in \mathbb{D}$ , let  $Q(\zeta) = f(\gamma\zeta)/\gamma$ , where  $\gamma \in (0, 1)$ . It is not difficult to know that  $Q(0) = |J_Q(0)| - \lambda = 0$ . By (3.8), for  $\zeta \in \mathbb{D}$ , we obtain

$$|Q(\zeta)| = \frac{|f(\gamma\zeta)|}{\gamma} \leq c_{\alpha}^{\frac{1}{p}} \|f\|_p \frac{(1 + \gamma)^{\frac{\alpha+1}{p}}}{\gamma(1 - \gamma)^{\frac{1}{p}}},$$

which gives that

$$|Q(\zeta)| \leq c_\alpha^{\frac{1}{p}} \|f\|_p \min_{0 < \gamma < 1} \mu(\gamma),$$

where

$$\mu(\gamma) = (1 + \gamma)^{\frac{\alpha+1}{p}} / \left[ \gamma(1 - \gamma)^{\frac{1}{p}} \right].$$

Let  $\gamma_0 \in (0, 1)$  satisfy

$$\mu(\gamma_0) = \min_{0 < \gamma < 1} \mu(\gamma).$$

By using Lemma 2, we observe that  $Q$  is univalent in  $\mathbb{D}_{\rho_0}$ , where  $\rho_0$  satisfies the following equation

$$\frac{\lambda}{M^*(2 + \alpha)} - \frac{4M^*\rho_0}{\pi} \left[ \frac{2 - \rho_0}{(1 - \rho_0)^2} + \frac{2\rho_0}{(1 - \rho_0)(1 - \rho_0^2)^2} \right] = 0,$$

where  $M^* = c_\alpha^{\frac{1}{p}} \|f\|_p \mu(\gamma_0)$ . Moreover,  $Q(\mathbb{D}_{\rho_0})$  contains a univalent disk  $\mathbb{D}_{R_0}$  with

$$R_0 \geq \frac{2\rho_0}{3} \left[ \frac{\lambda}{M^*(2 + \alpha)} - \frac{M^*\rho_0(2 - \rho_0)}{\pi(1 - \rho_0)^2} \right].$$

Hence  $f$  is univalent in  $\mathbb{D}_{\gamma_0\rho_0}$  and  $f(\mathbb{D}_{\gamma_0\rho_0})$  contains a univalent disk  $\mathbb{D}_{\gamma_0 R_0}$ . The proof of this theorem is complete.  $\square$

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